MULTIPLICATIVE STRUCTURES ON MOORE SPECTRA

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1. Introduction

In my talk today I am going to follow Robert Burklund's . The basic idea is not hard to state. He first constructs an obstruction theory for equipping quotients with multiplicative structures for an arbitrary stable ∞-category and then shows that we can actually show these obstructions vanish when we study them in $\mathrm{Syn}_{\mathbb{F}_2}$. This is in stark contrast to running the argument in Sp. Intuitively, one can think of $\mathrm{Syn}_{\mathbb{F}_2}$ as providing an extra grading that spreads classes out by remembering their Adams filtration. This extra dimension increases the sparsity of the homotopy groups, allowing us to conclude that the relevant obstructions vanish, whereas these groups are not trivial when only viewed spectra. He then uses the symmetric monoidal realization functor to transport the synthetic multiplicative structure to a classical one.

Warning 1.1. Robert uses the convention that filtrations are increasing in his paper. This conflicts both with my own previous lecture as well as most of the literature on the subject, minus Higher Algebra. I have attempted to carefully switch this back to the decreasing convention, but I may have made typographic errors. As such any indexing issues are almost certainly mine.

2. A Toy Example

In his paper, Robert begins with an example that I found mystifying for a long time. I am going to try to work it out in detail here. Fix $\mathcal{C}$ to be a presentably symmetric monoidal stable ∞-category with unit $1_{\mathcal{C}}$. Let $\mathcal{C}^{\mathrm{fil}}$ be its category of filtered objects, and τ the usual deformation parameter.

Let $\mathcal{I} \in \mathcal{C}$ be an arbitrary object equipped with a map $v : \mathcal{I} \to 1_{\mathcal{C}}$. There are two quotients to consider in this situation. The first is the usual cofiber in $\mathcal{C}$, which we will denote $1_{\mathcal{C}}/v$. There is then the $\mathcal{E}_n$-quotient defined as the pushout of

$$\begin{array}{ccc} 1_{\mathcal{C}}\{\mathcal{I}\} & \xrightarrow{v} & 1_{\mathcal{C}} \\ {\scriptstyle 0}\downarrow & & \downarrow \\ 1_{\mathcal{C}} & \longrightarrow & 1_{\mathcal{C}} /\!\!/^n v \end{array}$$

in $\mathrm{Alg}_{\mathbb{E}_n} \mathcal{C}$, where $1_{\mathcal{C}}\{\mathcal{I}\}$ is the free $\mathbb{E}_n$-algebra. In trying to construct an $\mathbb{E}_n$-algebra structure on $1_{\mathcal{C}}/v$, we might start with $1_{\mathcal{C}} /\!\!/^n v$ and attempt to attach $\mathbb{E}_n$-cells until we get a ring with underlying object $1_{\mathcal{C}}/v$.

And enhancement of this strategy is to first spread things out a bit. Let $1_{\mathcal{C}^{\mathrm{fil}}}$ be the unit in $\mathcal{C}^{\mathrm{fil}}$. It is the image of $1_{\mathcal{C}}$ under a (unique) symmetric monoidal left adjoint $\iota : \mathcal{C} \to \mathcal{C}^{\mathrm{fil}}$. We then have the composition:

$$\iota\mathcal{I}[-1] \to 1_{\mathcal{C}^{\mathrm{fil}}}[-1] \xrightarrow{\iota(v)[-1]} 1_{\mathcal{C}^{\mathrm{fil}}}[-1] \xrightarrow{\tau} 1_{\mathcal{C}^{\mathrm{fil}}}$$

which has levelwise depiction:

$$\begin{array}{ccccc}
\dots & \longrightarrow & \dots & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 1_{\mathcal{C}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I} & \xrightarrow[v]{} & 1_{\mathcal{C}} & \xrightarrow[\mathrm{id}]{} & 1_{\mathcal{C}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I} & \xrightarrow[v]{} & 1_{\mathcal{C}} & \xrightarrow[\mathrm{id}]{} & 1_{\mathcal{C}} \\
\downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & \dots & \longrightarrow & \dots
\end{array}$$

We will abuse notation to identify $\iota(v)$ and all of its shifts with v. Note that after inverting τ, we can recover $1_{\mathcal{C}} /\!\!/^1 v$ and $1_{\mathcal{C}}/v$. We can then depict $1_{\mathcal{C}^{\mathrm{fil}}}/\tau v$ as

$$\dots \longrightarrow 0 \longrightarrow 1_{\mathcal{C}} \longrightarrow 1_{\mathcal{C}}/v \longrightarrow 1_{\mathcal{C}}/v \longrightarrow \dots$$

Let us try to compute some of $1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/^1 \tau v$. To do so, we will study the effect of killing τ, i.e., taking the associated graded.

Lemma 2.1. There is an equivalence of $\mathbb{E}_1$-rings

$$\mathrm{gr}_* 1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/^1 \tau v \simeq 1_{\mathcal{C}^{\mathrm{gr}}}\{\Sigma\mathcal{I}(-1)\}$$

of objects in $\mathcal{C}^{\mathrm{gr}}$, where $\mathcal{I}(-1)$ is the graded object with L in degree -1 and 0 elsewhere.

Proof. Killing τ will preserve the defining pushout diagram

$$\begin{array}{ccc}
1_{\mathcal{C}^{\mathrm{fil}}}\{\iota\mathcal{I}[-1]\} & \xrightarrow{\tau v} & 1_{\mathcal{C}^{\mathrm{fil}}} \\
{\scriptstyle 0}\downarrow & & \downarrow \\
1_{\mathcal{C}^{\mathrm{fil}}} & \longrightarrow & 1_{\mathcal{C}} /\!\!/^1 v
\end{array}$$

and send it to the diagram

$$\begin{array}{ccc}
1_{\mathcal{C}^{\mathrm{gr}}}\{\mathcal{I}(-1)\} & \xrightarrow{0} & 1_{\mathcal{C}^{\mathrm{gr}}} \\
{\scriptstyle 0}\downarrow & & \downarrow \\
1_{\mathcal{C}^{\mathrm{gr}}} & \longrightarrow & \mathrm{gr}_* 1_{\mathcal{C}} /\!\!/^1 v
\end{array}$$

This diagram has another description. Namely, it can be reconstructed by taking the span $0 \leftarrow \mathcal{I}(-1) \rightarrow 0$, applying the free algebra functor, and taking the pushout. But the free algebra functor preserves colimits, so that $\mathrm{gr}_* 1_{\mathcal{C}} /\!\!/^1 v \simeq 1_{\mathcal{C}^{\mathrm{gr}}}\{\Sigma\mathcal{I}(-1)\}$. □

We now know that $1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/^1 \tau v$ is "built from" the graded pieces of the object $1_{\mathcal{C}^{\mathrm{gr}}}\{\Sigma\mathcal{I}(-1)\}$. It is worth investigating this object in low degrees. Recall that the free $\mathbb{E}_1$-algebra on a module M looks like

$$1\{M\} = \bigoplus_k M^{\otimes k+1}$$

on underlying objects. By putting L in degree 1 this allows us to view $1_{\mathcal{C}^{\mathrm{gr}}}\{\mathcal{I}(-1)\}$ as having graded pieces $\mathcal{I}^{\otimes k}$ in degree k. As a result, we know that $1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/^1 \tau v$ is built from the cells $\mathcal{I}^{\otimes k}$ placed in degree $-k$. It remains to understand how they are attached.

Consider again the defining pushout

$$\begin{array}{ccc} 1_{\mathcal{C}^{\mathrm{fil}}}\{\iota\mathcal{J}[-1]\} & \xrightarrow{\tau v} & 1_{\mathcal{C}^{\mathrm{fil}}} \\ {\scriptstyle 0}\downarrow & & \downarrow \\ 1_{\mathcal{C}^{\mathrm{fil}}} & \longrightarrow & 1_{\mathcal{C}} /\!\!/^{1} v \end{array}$$

We will only attempt to understand this quotient in low degrees.

Lemma 2.2. The degree -1 part of the defining pushout for $1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/^{1} \tau v$ is

$$\begin{array}{ccc} 1_{\mathcal{C}} \oplus \mathcal{J} & \xrightarrow{\mathrm{id}\oplus v} & 1_{\mathcal{C}} \\ {\scriptstyle 0}\downarrow & & \downarrow \\ 1_{\mathcal{C}} & \longrightarrow & 1_{\mathcal{C}}/v \end{array}$$

Proof. First we show that the degree -1 filtered part of $1_{\mathcal{C}^{\mathrm{fil}}}\{\mathcal{J}[-1]\}$ is $1_{\mathcal{C}} \oplus \mathcal{J}$. We need to compute the degree one part of $\bigoplus_k \iota(\mathcal{J})[-1]^{\otimes k}$. Note that $X[s] = X \otimes 1_{\mathcal{C}^{\mathrm{fil}}}[s]$ so that

$$\iota(\mathcal{J})[-1]^{\otimes k} \simeq \iota(\mathcal{J})^{\otimes k} \otimes 1_{\mathcal{C}^{\mathrm{fil}}}[-1]^{\otimes k} \simeq \iota(\mathcal{J}^{\otimes k})[-k]$$

As we can see, the only terms that will contribute to degree -1 are $\iota(\mathcal{J})[-1]^{\otimes 0}$ and $\iota(\mathcal{J})[-1]^{\otimes 1}$. That the top horizontal map is $\mathrm{id} \oplus v$ follows.

If we knew that the forgetful functor preserved pushouts, we would be done, and we apply the lemma below. It turns out that it is for reasons I will explain if asked but will not get into here. □

As a result, we know that $1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/^{1} \tau v$ looks like:

$$... \to 0 \to 1_{\mathcal{C}} \to 1_{\mathcal{C}}/v \to (?) \to ...$$

Through degree -1 this agrees with $1_{\mathcal{C}^{\mathrm{fil}}}/\tau v$. Our goal will be to attempt to force this higher filtered pieces to agree. The degree -2 part fits into a cofiber:

$$1_{\mathcal{C}}/v \to (?) \to \Sigma^2 \mathcal{J}^{\otimes 2}$$

and we would like to kill the contribution of the third term. To do so we need a lift:

$$\begin{array}{ccc} & & 1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/ \tau v \\ & \nearrow & \downarrow \\ \iota(\Sigma^2 \mathcal{J}^{\otimes 2})[-2] & \longrightarrow & (1_{\mathcal{C}^{\mathrm{fil}}} /\!\!/ \tau v)/\tau \end{array}$$

so that we might take a further $\mathbb{E}_1$-cofiber. The obstruction to doing so is a class:

$$\bar{Q}_1(v) \in [\Sigma^1 \mathcal{J}^{\otimes 2}, 1_{\mathcal{C}}/\tau]$$

which, if zero, allows us to equip $1_{\mathcal{C}}/\tau v$ with a unital multiplication as this would appear as the degree ≥ -2 part of the filtered object so-constructed.

3. Recollections on the $\mathbb{E}_n$-operads

Recall that one point of view on the $\mathbb{E}_n$-operads is that they are formed as a sequence of spaces $\mathbb{E}_n(k)$ with a natural Σ_k-action. The homotopy orbits of $\mathbb{E}_n(k)$ are given (up to homotopy) by the space of unordered configurations of k points in $\mathbb{R}^n$.

An advantage of the ∞-formalism is that spaces (∞-groupoids) can be viewed as a full subcategory of Cat_∞. As a result, given an operad $\mathcal{O}$ of spaces, one can define $\mathcal{O}$-algebra structures on ∞-categories in the same way as one does on spaces, i.e. such a structure is given by a series of structure maps:

$$\theta_k : \mathcal{O}(k) \times \mathcal{C}^{\times k} \to \mathcal{C}$$

which are all Σ_k-equivariant. We can use the diagonal $\mathcal{C} \to \mathcal{C}^{\times k}$ to restrict this to an equivariant functor

$$\mathcal{O}(k) \times \mathcal{C} \to \mathcal{C}$$

which has an adjoint pair

$$\mathcal{C} \to \mathrm{Fun}_{\Sigma_k}(\mathcal{O}(k), \mathcal{C})$$

Any functor on the right automatically factors as $\mathcal{O}(k)_{h\Sigma_k} \to \mathcal{C}$.

Definition 3.1. Let $\mathcal{C}$ be a presentably $\mathbb{E}_k$-monoidal ∞-category. Then we define $\mathbb{D}_k^n : \mathcal{C} \to \mathcal{C}$ to be the composition:

$$\mathcal{C} \to \mathrm{Fun}(\mathbb{E}_n(k)_{h\Sigma_k}, \mathcal{C}) \xrightarrow{\text{take limit}} \mathcal{C}.$$

This construction will be of essential important in phrasing the obstruction theory.

4. Obstruction Theory

Using material from his appendix (which I won't get into) Robert extends the ideas above as follows:

Lemma 4.1. Let $\mathcal{I} \in \mathcal{C}$. Then there exists a sequence of $\mathbb{E}_n$-rings in $\mathcal{C}^{\mathrm{gr}}$

$$1_{\mathcal{C}^{\mathrm{gr}}} = R_0 \xrightarrow{r_1} R_1 \xrightarrow{r_2} R_2 \xrightarrow{r_3} ... \to 1_{\mathcal{C}^{\mathrm{gr}}} \oplus \Sigma\mathcal{I}(-1)$$

converging to the trivial square zero extension of the unit by ΣX placed in degree -1. Moreover the maps r_k appear in pushouts of the form:

$$\begin{array}{ccc} 1_{\mathcal{C}^{\mathrm{gr}}}\{\Sigma^{-1-n}\mathbb{D}_k^n(\Sigma^{n+1}X(-1))\} & \longrightarrow & 1_{\mathcal{C}^{\mathrm{gr}}} \\ \downarrow & & \downarrow \\ R_k & \xrightarrow[r_k]{} & R_{k+1} \end{array}$$

and each R_k is equivalent to $1_{\mathcal{C}^{\mathrm{gr}}} \oplus \Sigma\mathcal{I}(-1)$ through degree $-k$.

Proposition 4.2. Given a map $v : \mathcal{I} \to 1_{\mathcal{C}}$ there exist inductively defined obstructions:

$$\Theta_k \in [1_{\mathcal{C}}\{\Sigma^{-2-n}\mathbb{D}_k^n(\Sigma^{n+1}\mathcal{I}), 1_{\mathcal{C}}/v]$$

for $k \geq 2$ allowing one to construct a sequence of $\mathbb{E}_n$-rings

$$\bar{R}_0 \to \bar{R}_1 \to ... \to 1_{\mathcal{C}}/v$$

which will converge to an $\mathbb{E}_n$-structure on $1_{\mathcal{C}^{\mathrm{gr}}}$. These all fit into an analogous pushout square.

Proof Sketch. The idea is to produce at each stage a filtered ring $\tilde{R}_k$ whose associated graded is R_k and which realize to $\hat{R}_k$. Given $\tilde{R}_{k-1}$, we produce $\tilde{R}_k$ by lifting

$$\begin{array}{ccc} & & \tilde{R}_{k-1} \\ & \nearrow & \downarrow \\ 1_{\mathcal{C}^{\mathrm{fil}}}\{\Sigma^{-1-n}\mathbb{D}_k^n(\Sigma^{n+1}X(-1))\} & \longrightarrow & R_{k-1} \end{array}$$

and pushing out in a square as in the theorem statement. One checks that the obstruction to doing so is the class listed. We then define $\bar{R}_k$ to be the realization of $\tilde{R}_k$. That the $\tilde{R}_k$ converge to $1_{\mathcal{C}^{\mathrm{fil}}}/\tau v$ is checked on associated and the convergence of the $\bar{R}_k$ follows. □

The final result we will need will increase the number of obstructions but in doing so make the groups we need to check simpler.

Lemma 4.3. There exists a resolution of $\mathcal{D}_n^k(Y)$ by finitely many copies of $\Sigma^{-c}Y$ where $0 \leq c \leq (n-1)(k-1)$.

Proof Sketch. This comes from the cellular filtration on the space of unordered configurations which models $\mathbb{E}_n(k)_{h\Sigma_k}$. □

Proposition 4.4. There exist refinements of the obstructions θ_k, which we denote $\theta_{k,\alpha}$ of the form:

$$\theta_{k,\alpha} \in [\Sigma^{-2-n-c_\alpha}(\Sigma^{n+1}X)^{\otimes k}, 1_{\mathcal{C}}/v]$$

where $c_\alpha \leq (n-1)(k-1)$. For $n = 1$ this simplifies to obstructions

$$\theta_k \in [\Sigma^{-3}(\Sigma^2 X)^{\otimes k}, 1_{\mathcal{C}}/v]$$

as $\mathbb{E}_1(k)_{h\Sigma_k}$ is a point.

5. Moore Spectra

We now will apply the obstruction theory to prove Robert's results on multiplicative structures on Moore spectra. We not be applying the theory to the category Sp. Although we could do so, the obstructions live in groups that are not trivial in Sp. Instead, we will provide a multiplicative structure on a lift of the Moore spectra to $\mathrm{Syn}_{\mathbb{F}_2}$ and then use the symmetric monoidal realization functor to provide the desired structure on the classical spectrum.

Convention 5.1. We will grade synthetic spectra by $\mathbb{S}^{t,f}$ where t is the topological degree, i.e. $\Re(\mathbb{S}^{t,f}) \simeq \mathbb{S}^t$ and f denoted Adams filtration.

Recall that there is a fully faithful functor $\nu : \mathrm{Sp} \to \mathrm{Syn}_{\mathbb{F}_2}$ which is a section of $\Re : \mathrm{Syn}_{\mathbb{F}_2} \to \mathrm{Sp}$. Given any map of spectra $f : X \to Y$ we may lift it by taking $\nu(f)$. This will not be a unique lift, however. Since realization corresponds to inverting τ, we see that any τ-multipliple or even τ-division of $\nu(f)$. This presents a nontrivial choice even in the case f is the map $2 : \mathbb{S} \to \mathbb{S}$. We will choose the class:

$$\tilde{2} : \mathbb{S}^{0,1} \to \mathbb{S}^{0,0}$$

which is uniquely defined as $\frac{\nu(2)}{\tau}$.[1]

Theorem 5.2. *There exists an $\mathbb{E}_1$ structure on $\mathbb{S}/2^3$.*

Proof Sketch. Lifting to the synthetic world, we study the map of synthetic spectra

$$\tilde{2}^3 : \mathbb{S}^{0,3} \to \mathbb{S}$$

whose obstructions are classes $\theta_k \in [\Sigma^{-3,3}(\mathbb{S}^{2,1})^{\otimes k}, \mathbb{S}^{0,0}/\tilde{2}^3]$ a synthetic refinement of Adam's $\mathbb{E}_2$ vanishling line shows that all of these classes live in 0 groups as their filtration is too high. □

Nearly identical proofs show the following claims:

Theorem 5.3. *$\mathbb{S}/2^q$ admits an $\mathbb{E}_n$-multiplication as soon as $q \geq \frac{3}{2}(n+1)$*

6. The General Case

It is well known that I cannot resist giving the general version of a result. I will make this brief. Robert shows that in general we can construct a deformation for $\mathcal{C}$ which plays the role $\mathrm{Syn}_{\mathbb{F}_2}$ plays above using machinery of Piotr and Irakli. He uses this to prove:

Theorem 6.1. *Suppose $1_{\mathcal{C}}/v$ is a quotient admiting a right unital multiplication*[2]*. Then there is a compatible sequence of $\mathbb{E}_n$-algebras*

$$\ldots \to 1_{\mathcal{C}}/v^{n+3} \to 1_{\mathcal{C}}/v^{n+2} \to 1_{\mathcal{C}}/v^{n+1}$$

He uses this more powerful theorem to prove two important corollaries.

Corollary 6.2. *For $p > 2$ there is a compatible $\mathbb{E}_n$-multiplication on $\mathbb{S}/p^{n+1}$.*

Corollary 6.3. *For all p, h, n there is a height h generalized Moore spectrum with $\mathbb{E}_n$-multiplication of the form*

$$\mathbb{S}/(p^{i_0}, \ldots, v_{h-1}^{i_{h-1}})$$

for choices of exponents i_j sufficiently large.[3]

[1] Every map $\nu(f)$ is exactly as τ-divisible as its Adams filtration.

[2] One can use the deformation theory we spoke of at the beginning to attempt to estbalish this

[3] The size of the i_j depends not only on the obstruction theory but also on the existence results for v_j-self maps.