MULTIPLICATIVE STRUCTURES ON MOORE SPECTRA

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1. Introduction

In my talk today I am going to be discussing the recent preprint of Christian Carrick on Chromatic Defect https://arxiv.org/abs/2402.17519. The idea is to quantify by an nonnegative, possibly infinite integer n how far a spectrum is from being complex orientable. We will then discuss how this question ends up being closely related to the existence of Wood type splittings. Recall wood's theorem:

Theorem 1.1 (Wood)**.** $\mathrm{ko} \otimes C\eta \simeq \mathrm{ku}$

One interesting observation is that while ku is complex-orientable, the spectrum ko is not. This idea will be generalized to define Wood-type spectra later in the talk, and we will explain how it is related to complex orientations in general.

2. Orientations

In this talk, all rings and models should be understood in the weakest possible sense, i.e., rings E as monoids in $h\mathrm{Sp}$ and modules M as spectra for which there exists a diagram

$$M \xrightarrow{\text{unit}} E \otimes M \to M$$

factoring the identity on M. Let $\iota_n : \mathbf{CP}^n \to \mathbf{CP}^\infty$ be the standard inclusion. Recall that a homotopy ring spectrum E is said to be complex-orientable if the induced map

$$E^2(\mathbf{CP}^\infty) \xrightarrow{\iota_1} E^2(\mathbf{CP}^1)$$

is surjective. Because $\mathbf{CP}^1 \simeq S^2$, the right hand side is canonically isomorphic to $\pi_0 E$. An orientation of E is a choice of preimage for 1 under ι_1. It is an important theorem in the study of chromatic homotopy theory that orientations are in bijection with MU-algebra structures $\mathrm{MU} \to E$.

It will be convenient to expand the defintion of complex-orientability to work with spectra X which are not necessarily rings. Recall that one construction of $\mathbf{CP}^n$ is to equip S^{2n+1} with the circle action $\mathbf{T}$ and quotient $\mathbf{CP}^n \simeq S^{2n+1}/\mathbf{T}$. Write σ_k for the quotient map $S^{2k+1} \to \mathbf{CP}^k$. The cofiber of σ_k is $\mathbf{CP}^{k+1}$, so we may think of σ_k as the top-cell attaching map for $\mathbf{CP}^{k+1}$. Note that

$$\sigma_1 : S^3 \to \mathbf{CP}^1 \simeq S^2$$

is the map $\Sigma^2\eta$, identifying $C\eta \simeq \Sigma^{-2}\mathbf{CP}^2$.

Definition 2.1. A spectrum Y is complex orientable if $Y \otimes \sigma_k \simeq 0$ for all $k \geqslant 1$. In effect, we ask that Y kills all the attaching maps for $\mathbf{CP}^\infty$.

Remark 2.2. An immediate consequence of this definition is that it is closed under retracts and tensoring with arbitrary spectra.

Proposition 2.3. If E is a ring spectrum then both definitions of complex orientable coincide.

Proof. First suppose E kills the attaching maps. Then we have to show that $\mathbf{S} \to E$ factors over $\Sigma^{-2}\mathbf{CP}^\infty$. But the assumption gives a splitting and therefore a composition:

$$\Sigma^{-2}\mathbf{CP}^\infty \to E \otimes \Sigma^{-2}\mathbf{CP}^\infty \simeq \bigoplus_{k \geqslant 0} \Sigma^{2k} E \to E$$

in the opposite direction, given an extension over $\Sigma^{-2}\mathbf{CP}^\infty$ of the unit, we get a class in $x \in \pi_{-2} F(\mathbf{CP}^k_+, E)$ and can define

$$f_k : \bigoplus_{0 \leqslant n \leqslant k} \Sigma^{-2n} E \to F(\mathbf{CP}^k_+, E)$$

whose nth component is $\Sigma^{-2n}E \xrightarrow{x^n} E \otimes F(\mathbf{CP}^k_+, E) \to F(\mathbf{CP}^k_+, E)$. When we filter the left hand side in n and the right hand side via the cellular filtration on $\mathbf{CP}^k$, we get a map of filtrations and one can check that it induces an isomorphism on associated gradeds. This implies that $E \otimes \mathbf{CP}^{k-1} \to E \otimes \mathbf{CP}^k$ splits, so that the attaching map must be killed. □

An upshot of this definition is that it also extends the comparison with MU:

Proposition 2.4. If Y is complex orientable (and not necessarily a ring) if and only if it is an MU-module.

We will return to the above and sketch its proof later.

Corollary 2.5. If Y is complex orientable the Adams-Novikov for Y collapses onto the 0-line.

A great many spectra of interest are not, themselves, complex orientable.

3. Review of $X(n)$ and $T(n)$

Recall that there is an equivalence $\Omega SU \simeq BU$. One way to define MU is as the Thom spectrum of the map $\Omega SU \to BU$, which is a map of infinite loop spaces, granting MU the structure of an $\mathbf{E}_\infty$-ring. Moreover, we have a filtration of the left hand side

$$* \simeq \Omega SU(1) \to \Omega SU(2) \to ... \to SU$$

by 2-fold loop spaces with 2-fold loop maps. Putting $X(n)$ for the Thom spectrum of $\Omega SU(n) \to BU$ we get a filtration by $\mathbf{E}_2$-rings

$$\mathbf{S} = X(1) \to X(2) \to ... \to \operatorname{colim} X(n) \simeq \mathrm{MU}$$

whose original claim to fame is their integral role in the Nilpotence Theorem of DHS. The central idea is that each map $X(n) \to X(n+1)$ detects nilpotence, and one descends this tower to the sphere.[1] Note that $X \otimes \mathrm{MU}$ is an MU-module by construction, and if Y is itself an MU-module, then $Y \otimes X(n)$ is as well for all n. As a result, one can view the attempt to orient Y as an attempt to descend the $X(n)$-tower similarly to the Nilpotent Theorem.

4. Chromatic Defect and Wood Types

Proposition 4.1. A spectrum Y kills the attaching maps σ_k for $1 \leqslant k \leqslant n-1$ if and only if Y is an $X(n)$-module, where $1 \leqslant n \leqslant \infty$.

Proof sketch. We first assume $Y = E$ is a ring satisyfing (2). As inpute we take the isomorphisms:

$$H_*(X(n); \mathbf{Z}) = \mathbf{Z}[b_1, b_2, ..., b_{n-1}]$$

From here, we consider the map $\Sigma^{-2}\mathbf{CP}^n \to X(n)$ which induces on AHSS E_2 pages the map

$$E_* \otimes H_*(\Sigma^{-2}\mathbf{CP}^n) \to E_* \otimes H_*(X(n))$$

this spectral sequence ends up collapsing, allowing us to identify $E \otimes X(n)$ with $E\{M\}$ where M is a monomial basis for the ring $\mathbf{Z}[b_1, ..., b_{n-1}]$. The action $X(n) \otimes E \to E$ is the projection corresponding to 1.

For the non-ring case we observe that the endomorphism ring of Y will still kill the attaching maps, so that $\mathrm{End}(Y)$ is an MU-module, and Y is certainly an $\mathrm{End}(Y)$-module. Conversely, if Y is an $X(n)$-module, then $Y \otimes \sigma_k$ is a retract of $Y \otimes X(n) \otimes \sigma_k$, so the maps must be killed as $X(n)$ kills them. □

In particular, the spectrum $Y \otimes X(n)$ is an $X(n)$ module, and therefore kills the appropriate range of attaching maps. From this point of view, we can think of the functor $- \otimes X(n)$ as forcibly killing finitely many of the attaching maps. We can then ask whether, after killing finitely many of them, the rest die for free.

Definition 4.2. A spectrum Y is said to have *chromatic defect* $\leqslant n$ if $Y \otimes X(n)$ is complex orientable. It has chromatic defect n if this is the least such n. We write $\Phi(Y)$ for the chromatic defect of Y.

Note that if $Y \otimes X(n)$ is complex orientable and $m \geqslant n$ then $Y \otimes X(m)$ is also complex orientable, which is immediate from the perspective of modules. Indeed, there is a stronger result:

Lemma 4.3. If $R \to S$ is a ring map then $\Phi(R) \geqslant \Phi(S)$.

[1] In a project that I will eventually finish, I show that one can bound the lossage of exponent that occurs at each stage.

Proof. If $R \otimes X(n)$ is complex orientable if and only if it is an MU-module. But then

$$S \otimes X(n) \simeq S \otimes_R (R \otimes X(n))$$

is again an MU-module. □

Proposition 4.4. A spectrum Y has chromatic defect $\leqslant n$ if and only if $Y \otimes \sigma_k \simeq 0$ for all $1 \leqslant k \leqslant n-1$.

Remark 4.5. One interpretation of the above is that the sphere is the "least" complex orientable ring, as every other ring will accept a map $\mathbf{S} \to R$. Numerically, this is seen in the fact that $\Phi(\mathbf{S}) = \infty$ as the sphere kills no attaching maps.

Example 4.6. We will prove the following as we go:
(1) $\Phi(\mathbf{S}) = \infty$
(2) $\Phi(\mathrm{ko}) = 2$
(3) $\Phi(\mathrm{tmf}) = 4$.
and $\Phi(E) = 1$ if and only if E itself is complex orientable.

For the remainder of this talk, it will be helpful to work p-locally. Recall that MU splits with summand BP after doing so. Heuristically, the $X(n)$ spectra can be viewed as adding in each of the polynomial generators of MU, and indeed $X(n) \to \mathrm{MU}$ is an equivalence in the appropriate range. Then BP has only a subset of these generators in the v_n, so one might expect that there is a "shorter" filtration at a prime. Indeed, there are spectra $T(n) := X(p^n)[\epsilon^{-1}]$ where ϵ is the idempotent splitting BP off of MU. They are only known to be $\mathbf{E}_1$.

Definition 4.7. A spectrum Y has p-local chromatic defect $\leqslant n$ of $Y \otimes T(n)$ is complex orientable. We write $\Phi_p(Y)$ for the p-local version.

Lemma 4.8. When E is p-local $\Phi_p(Y) = \lfloor \log_p \Phi(Y) \rfloor$.

We now turn to the concept of Wood type spectra. Recall that a finite spectrum F is said to be BP-projective if $\mathrm{BP}_* F$ is projective over BP_*. This is equivalent to asking that it be finite free.

Definition 4.9. A p-local spectrum Y is said to be of Wood Type if there exists a finite BP-projective spectrum F such that $Y \otimes F$ is complex orientable.

Example 4.10. Note that $\mathrm{BP}_* C\eta$ is finite free, so that the Wood theorem $\mathrm{ko} \otimes C\eta \simeq \mathrm{ku}$ exhibits ko as Wood-type, justifying the defintion.

The notion of Wood-type is closely related to chromatic defect via:

Proposition 4.11. Every finite BP-free F is a finite $T(n)$-free for some $n < \infty$.

Proof sketch. As spectra, $BP \otimes F = \mathrm{BP}[A]$ for some finite index set A. Choosing maps $S^\alpha \to \mathrm{BP} \otimes F$ for $\alpha \in A$, we see that the connectivity of $T(n) \to \mathrm{BP}$ allows us to choose $n >> 0$ such that all of the generators factor. Tensoring with BP over $T(n)$ this becomes an equivalent, and was therefore always an equivalent as BP is free over $T(n)$. □

Corollary 4.12. If Y is Wood-type then it has finite chromatic defect.

Proof. Let F be a finite $T(n)$-free spectrum witnessing the Wood-type-ness of Y. Then $Y \otimes T(n)$ is a summand of $Y \otimes T(n) \otimes F$ and is therefore an MU-module. □

5. Computing some Chromatic Defects

There turns out to be a nice obstruction theory for chromatic defects.

Construction 5.1. Because $T(n)$ is an $X(p^{n+1}-1)$-module, there is a splitting:

$$T(n) \otimes \Sigma^{-2} \mathbf{CP}^{p^{n+1}-1} \simeq T(n)\{b_0, ..., b_{p^{n+1}-1}\}$$

in degrees less than $2p^{n+1}-3$ the spectrum $T(n)$ looks like BP and therefore its homotopy groups vanish in even degrees. As a result, the attaching map:

$$\sigma_{p^{n+1}-1} : \mathbf{S}^{2p^{n+1}-3} \to \Sigma^{-2} \mathbf{CP}^{p^{n+1}-1}$$

has nonzero component only at the b_0 copy of $T(n)$. Denote by $\chi_{n+1} \in \pi_{2p^{n+1}-3} T(n)$ the projection.

Proposition 5.2. The class χ_{n+1} generates $\pi_{2p^{n+1}-3}T(n) \cong \mathbf{Z}/p$.

Theorem 5.3 (Beardsley)**.** If $p^n \leqslant m < p^{n+1}$ then $X(m+1) = X(m)[b_m]$ is the free $\mathbf{E}_1$-$X(m)$-algebra on a degree $2m$ generator. If $m = p^{n+1} - 1$ then $X(m+1)$ is the free $\mathbf{E}_1$-$X(n)$-algebra with a nullhomotopy of χ_{n+1}.

Proposition 5.4. For E a p-local ring spectrum, $\Phi_p(E) \leqslant n$ if and only if χ_{m+1} has trivial Hurewicz image in $E_*T(m)$ for all $m \geqslant n$.

Proof. $E \otimes T(m)$ is a $T(m+1)$-module if and only if

$$\sigma_{p^{m+1}-1} : \Sigma^{2p^{m+1}-3} E \otimes T(m) \to E \otimes T(m) \otimes \Sigma^{-2}\mathbf{CP}^{p^{m+1}-1}$$

is null. The domain splits as before, so it suffices to prove $E \otimes \chi_{m+1}$ is null. But because E is a ring, it suffices to check the condition in the proposition. □

5.1. Finite Spectra.

Theorem 5.5. Suppose that k, m are finite positive integers chosen such that $p^n \leqslant m < p^{n+1} \leqslant m + k$. Then there are no nontrivial compact $X(m)$-modules which are also $X(m+k)$-modules.

Corollary 5.6. For any finite spectrum F, we have $\Phi_p(F) = \Phi(F) = \infty$.

Proof. Choose p so that $F_{(p)}$ is nontrivial and drop the p-localization from the notation. Assume that $\Phi_p(F) = m < \infty$. Then note that $X(m) \otimes F$ is a compact $X(m)$-module, but then by assumption it is an MU-module, and therefore an $X(m+k)$ module for all k. □

Notation 5.7. Write $\mathcal{A}_*^{X(n)}$ for the relative dual Steenrod algebra $\pi_*(\mathbf{F}_p \otimes_{X(n)} \mathbf{F}_p)$ and write $H_*^{X(n)}(-)$ for the relative homology functor $\pi_*(- \otimes_{X(n)} \mathbf{F}_p)$. Note that the latter is a comodule over the former.

Remark 5.8. Note that there is a map $\mathcal{A}_* \to \mathcal{A}_*^{X(n)}$ coming from the spectral map:

$$\mathbf{F}_p \otimes \mathbf{F}_p \to \mathbf{F}_p \otimes_{X(n)} \mathbf{F}_p$$

induced by the pair of $\mathbf{S}$-linear maps including $\mathbf{F}_p$ into each side of the relative product.

We take the following lemmas as input in proving Theorem 5.5.

Lemma 5.9. If $p^n \leqslant m < p^{n+1}$ then the map $\mathcal{A}_* \to \mathcal{A}_*^{X(n)}$ sends $\zeta_{n+1}^{p^k}$ to a nonzero coalgebra primitive for each k when $p > 2$ and the same is true for $\zeta_{n+1}^{2^{k+1}}$ at $p = 2$.

Lemma 5.10. When F is a compact $X(m)$-module, $\mathrm{End}_{X(m)}(F)$ is a compact $X(m)$-module.

Lemma 5.11. When F is a compact $X(m)$-module, $H_*^{X(m)}X(m)$ is finitely generated.

Proof of Theorem 5.5. Recall that we have fixed $p^n \leqslant m < p^{n+1}$. Let F be a compact $X(m)$-module which we assume also admits the structure of an $X(m+k)$ module where $m + k > p^{n+1}$. By Beardsley's theorem, $\chi_{n+1} \in \pi_* X(m+k)$ is null, so that the map also is null in $\mathrm{End}_{X(m)}(F)$. As a result, we get a map

$$f : X(p^{n+1}) \to \mathrm{End}_{X(m)}(F)$$

via the universal property. Write E for the endomorphism ring on the right. The relative homology $H_*^{X(m)}(E)$ is bounded above by finite generation and we have $\mathcal{A}_*^{X(m)}$-comodule maps:

$$H_*T(n+1) \to H_*(X(p^{n+1})) \to H_*^{X(m)}(X(p^{n+1})) \to H_*^{X(m)}(E).$$

On the first term above, we have the coaction $\Psi(\zeta_{n+1}^{p^k}) = 1 \otimes \zeta_{n+1}^{p^k} + \zeta_{n+1}^{p^k} \otimes 1$ is primitive. As a result, the composite sends $\zeta_{n+1}^{p^k}$ to a nonzero element for all primes and $k > 1$, contradicting boundedness. □

5.2. **fp-Spectra.** Recall the following definition:

Definition 5.12. A p-complete bounded-below spectrum E is said to be fp if it satisfies any of the following equivalent conditions:

(1) There exists finite p-local F such that $\pi_* E \otimes F$ is finite.
(2) There exists a finite p-local F such that $E \otimes F$ splits as finitely many copies of $\mathbf{F}_p$.
(3) There exists a finite $\mathcal{A}(n)_*$-comodule M such that $H_*(E, \mathbf{F}_p) \cong \mathcal{A}_* \,\square_{\mathcal{A}(n)}\, M$.
(4) $H^*(E, \mathbf{F}_p)$ is finitely presented over the Steenrod algebra.

The third term in particular provides nice change-of-rings isomorphisms for the E_2-page of the Adams Spectral sequence:

$$\mathrm{Ext}_{\mathcal{A}}(\mathbf{F}_p, \mathcal{A} \,\square_{\mathcal{A}(n)_*}\, M) \cong \mathrm{Ext}_{\mathcal{A}(n)}(\mathbf{F}_p, M)$$

Moreover, if M is a $\mathcal{A}$-comodule which is cofree as a $\mathcal{P}(n-1)_*$-comodule $M \cong \mathcal{P}(n-1) \otimes V$, then

$$M \cong \mathcal{A}(n)_* \,\square_{\mathcal{E}(n)_*}\, V$$

leading to a second change of rings:

$$\mathrm{Ext}_{\mathcal{A}(n)_*}(\mathbf{F}_p, M) \cong \mathrm{Ext}_{\mathcal{E}(n)_*}(\mathbf{F}_p, V)$$

Leveraging these provides the following:

Proposition 5.13. If E is fp with $H_*(E, \mathbf{F}_p) \cong \mathcal{A}_* \,\square_{\mathcal{A}(n)}\, M$ then

$$H_*(E \otimes T(n)) \cong H_* E \otimes \mathcal{P}(n-1) \otimes \mathbf{F}_p[t_1^{2^n}, t_2^{2^{n-1}}, ..., t_n^2] \cong (\mathcal{A}_* \,\square_{\mathcal{E}(n)}\, M) \otimes \mathbf{F}_p[t_1^{2^n}, t_2^{2^{n-1}}, ..., t_n^2]$$

and the $\mathbf{E}_2$ page of the Adams spectral sequence for $E \otimes T(n)$ is

$$\mathrm{Ext}_{\mathcal{E}(n)_*}(\mathbf{F}_p, M) \otimes \mathbf{F}_p[t_1^{2^n}, t_2^{2^{n-1}}, ..., t_n^2]$$

Corollary 5.14. If E is an fp ring spectrum with presentation as above, then $\Phi_p(E) \leqslant n$ if

$$\mathrm{Ext}_{\mathcal{E}(n)}^{s, 2p^{m+1}-3-2(p-1)*+s}(\mathbf{F}_p, M) = 0$$

for all $s \geqslant 2$ and $m \geqslant n$. In particular, it holds if this Ext is concentrated in even stems $t - s$.

Example 5.15. One can check that the above applies to ko and tmf to show that the former has chromatic defect 2 and the latter has chromatic defect 4.

5.3. **Wood-type fp spectra.**

Definition 5.16. We say that a spectrum E is algebraicly Wood type if there exists a finite even comodule P such that $H_* E \otimes P$ is an $H_*\mathrm{MU}$-module.

Proposition 5.17. Every fp spectrum E with $H_* E \cong \mathcal{A}_* \,\square_{\mathcal{A}(n)_*}\, M$ with finite chromatic defect is algebraicly wood type.

Proof. Omitting the chromatic defect assumption, we always have an isomorphism

$$H_* E \otimes \mathcal{P}(n-1)_* \cong \mathcal{A}_* \,\square_{\mathcal{E}(n)_*}\, M$$

and when E has finite chromatic defect, $H_*(E \otimes T(m))$ is an $H_*\mathrm{MU}$ module. However, by the computation of $H_*(E \otimes T(m))$, the left hand side is a retract and therefore itself an $H_*\mathrm{MU}$-module. □

Theorem 5.18. If E is an fp ring with finite chromatic defect such that the Adams spectral sequence for $\mathrm{BP} \otimes E$ collapses at E_2, then E is Wood-type.

We take as input the lemma:

Lemma 5.19. There exist finite spectra F which are retracts of $(\mathbf{CP}^{p^n})^{\otimes N}$ for N large enough with the property that $H^*(F; \mathbf{F}_p)$ is a free $\mathcal{P}(n-1)$-module. Such an F is necessarily finite BP-projective.

Proof sketch of the theorem. Carrick shows that we can lift the $\mathcal{P}(n-1)$ to BP-projective spectra, so that the claim really relies on lifting the identification

$$H_*(E \otimes T(m)) \cong \mathcal{P}(n-1) \otimes \mathbf{F}_p[t_1^{2^n}, t_2^{2^{n-1}}, ..., t_n^2]$$

to a spectrum-level splitting. This relies on lifting the classes $t_i^{2^{n-i+1}}$ along the map $\nu(E \otimes T(n)) \to \nu(E \otimes T(n))/\tau$. This can be done when the Adams-spectral sequence for $E \otimes T(n)$ collapses at E_2 as this implies that there was no τ-torsion to begin with. Now choose F as in the lemma with basis for its homology $\{b_1, ..., b_k\}$. It is possible to fix a map:

$$\iota : F \to T(n)\{b_1, ..., b_k\}$$

which on homology

$$\bigoplus_{1 \leqslant i \leqslant k} \Sigma^* \mathcal{P}(n-1) \to \bigoplus_{1 \leqslant i \leqslant k} \mathcal{P}(n-1) \otimes \mathbf{F}_p\{t_1^{2^n}, t_2^{2^{n-1}}, ..., t_n^2\} \otimes \mathbf{F}_p\{b_i\}$$

sends b_i to b_i. We can extend this over

$$E \otimes F[t_1^{2^n}, t_2^{2^{n-1}}, ..., t_n^2] \to E \otimes T(n)\{b_1, ..., b_k\}$$

which is an equivalence. As a result, $E \otimes F$ is a retract of $E \otimes T(n)$ and is therefore complex orientable. □

Combining previous results:

Corollary 5.20. If E is an fp ring such that $H_* E \cong \mathcal{A}_* \,\square_{\mathcal{A}(n)_*}\, M$ and such that $\mathrm{Ext}_{\mathcal{E}(n)}(\mathbf{F}_p, M)$ is concentrated in even stems then E is Wood type with $\Phi_p(E) \leqslant n$.

6. Some Other Results

It would not do the paper justice to only present the computations above, so here we list them without proof.

Theorem 6.1. Here are some computations in the rest of the paper:

(1) $\Phi(j) = \infty$
(2) $\Phi(E_{\mathbf{R}}(n)^{hC_2}) = 2^n$ where $E_{\mathbf{R}}$ is a C_2-lift of Morava E-Theory.
(3) $\Phi(EO_n(G)) = p^N(G)$ where $EO_n(G) = E(n)^{hG}$ for G a finite subgroup of the Morava stabilizer group and $N(G)$ is a certain function in terms of the group.

And one last application:

Theorem 6.2. For E a Wood-type spectrum, the ANSS can be extended to a full plane spectral sequence using the duals of the finite BP-projective F in the definition. This **Z**-ANSS enjoys the following properties:

(1) The natural map $ANSS(E) \to \mathbf{Z} - ANSS(E)$ is an isomorphism on E_2 is positive filtration and a surjection in filtration 0.
(2) Along the same map there is a $1-1$ correspondce of differentials whose source has nonnegative filtration.
(3) The **Z**-ANSS converges to 0.

7. Some Synthetic Remarks

I am going to emphasize the point of view of $\mathbf{F}_p$-synthetic spectra in this section of the talk, which Carrick remarks upon but does not flesh out entirely. Any errors introduced are my own. In particular, claims with asterisks do not appear in the original paper.

I will only use $\mathbf{F}_p$-synthetic spectra, and will therefore denote the category Syn. Let $\nu : \mathrm{Sp} \to \mathrm{Syn}$ denote the synthetic analog. Many of the following definitions and claims are only implicit in the Carrick paper, and any errors introduced are my own. Recall importantly that $\mathrm{Mod}(C\tau) \simeq \mathrm{Stable}(\mathcal{A}_*)$ and that $\nu E/\tau$ is identified with $H_* E$ in the latter category.

Definition 7.1. A synthetic spectrum E is synthetically orientable if $- \otimes \nu E$ kills all of the synthetic attaching maps

$$\sigma_k^{syn} : \mathbf{S}^{2k+1,1} \to \nu\mathbf{CP}^k$$

or, equivalently, if it is a $\nu\mathrm{MU}$-module by arguments analogous to those in Sp.

Definition 7.2. A synthetic spectrum E is has synthetic chromatic defect $\leqslant n$ if $\nu(E \otimes X(n)) \simeq \nu(E) \otimes \nu X(n)$ is complex orientable.

Definition 7.3. A synthetic spectrum E is Wood type if there is a compact $\nu\mathrm{BP}$-projective F such that $E \otimes F$ is complex orientable.

Remark 7.4. The class σ_k^{syn} is not $\nu(\sigma_k)$, but satisfies $\tau\sigma_k^{syn} = \nu(\sigma_k)$. One should think of the former as representing a class in filtration 1 of E_2 which detects σ_k. Importantly, the former class is not τ-divisible, making the following definitions more sensible.

Remark 7.5. It is easy to see that if E has finite synthetic chromatic defect, then it will have at least the same chromatic defect as a spectrum, and likewise for orientability. Wood type is less clear to me.

Remark 7.6. If E is synthetically wood type, then it is automatically algebraicly wood type in the sense above, after identifying $C\tau$-modules with $\mathrm{Stable}(\mathcal{A}_*)$.

8. Some further questions

8.1. **Non-synthetic Questions.** All of the following are posed by Carrick in the original paper. Note that for a commutative ring spectrum E there are a number of chromatic numerical invariants we can ask about:

- The chromatic height of E,
- The BP-nilpotence exponent of E,
- The chromatic defect of E,
- For E Wood-type, the minimal n such that there exists a finite $X(n)$-projective F such that $E \otimes F$ is complex orientable.
- The orientation order of Bhattacharya-Chatham:

$$\Theta(E) = \min\{n \geqslant 1 \mid \xi^{\oplus n} \text{ is } E \text{ orientable.}\}$$

In the above, ξ is the tautological line bundle on $\mathbf{CP}^{\infty}$ and the A-nilpotence exponent of M is the least n for which $\bar{A}^{\otimes N} \otimes M \to M$ is null in the standard Adams tower.

The relations between the above are largely open. I think it would be interesting to related any of them to eachother. Carrick proposes, for example:

Question 8.1. Is $\Phi(E)$ a lower bound for the BP-nilpotence exponent of E?

Question 8.2. Does there exist a spectrum with finite chromatic defect which is not Wood type?

Question 8.3. How does chromatic defect interact with Red Shift? Note that strong forms of (unproven) Red shift assert that fp type n spectra are sent to fp type $n+1$ spectra by K-theory, where the type of an fp spectrum is defined to be the thick subcategory of finite spectra such that $|E_*V| \leqslant \infty$.

8.2. **Synthetic Questions.**

Question 8.4. If νE is synthetically Wood type, is E Wood type? This would be nice to know, as it sets up further questions about when algebraicly Wood type spectra are Wood type. Note that this will essentially depend on whether $\Re$ sends $\nu\mathrm{BP}$-projectives to BP-projectives.

Question 8.5. Is the synthetic chromatic defect of νE always the same as the chromatic defect of E? It is not hard to see that if we phrase the question in terms of $\nu(\sigma_k)$, the answer is yes. However, Carrick suggests using the class in degree $(2k+1, 1)$ which can be thought of as $\frac{\nu\sigma_k}{\tau}$ since it satisfies the implicit relation. Therefore, the only way this statement can fail to be true is if σ_k^{syn} is nontrivial τ-torsion, i.e., it is killed by a d_2 in the Adams spectral sequence.

Along a completely different route, it seems similarly interesting to think about Wood type spectra and chromatic defect from a BP-synthetic (or motivic) perspective. Indeed, the functor ν_{BP} is symmetric monoidal when one of the two inputs is finite BP-projective, allowing Wood-type-ness to be easily imported to the synthetic world. Moreover, $\mathbf{CP}^k$ has a much simpler ANSS than ASS, which could potentially make the study of the attaching maps more amenable. Here is one potential question:

Question 8.6. A finite BP-nilpotence exponent of a ring E induces a horizontal vanishing line in its ANSS, which corresponds to a vanishing region in $\pi_{*,*}\nu E$. Does an MU-synthetic approach to chromatic defect allow us to produce a bound?