SYMMETRIC MONOIDAL ALGEBRAICITY OF $L_n\mathrm{Sp}$ AT LARGE PRIMES

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CONTENTS

1. Introduction

The main reference for this talk is Shaul Barkan's paper on symmetric monoidal algebraicity. Specifically his paper's novelty is making the identification respect monoidal structures. Previous results, including the equivalence at the levels of categories, were already known by others. Shaul does a great job of reviewing this history, I won't repeat it here.

2. Motivation and Connections to the Seminar

Fix a prime p. Let X be a p-local spectrum. A common approach to attempting to understand X is through its Adams-Novikov spectral sequence:

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*X) \Rightarrow \pi_*X$$

The Hopf Algebroid (BP_*, BP_*BP) may be viewed as a stack $\mathcal{M}_{\mathrm{fg},(p)}$ which represents p-typical formal group laws. As a result, the ANSS for the sphere may be rewritten as:

$$H^*(\mathcal{M}_{\mathrm{fg},(p)}, \omega^{\otimes *}) \Rightarrow \pi_*\mathbb{S}_{(p)}$$

and this can be philosophically interpretted as an attempt to transfer information from algebraic geometry (via the theory of stacks) to stable homotopy theory. This translation is facilitated by the differentials in the spectral sequence, which are of course highly nontrivial.

This point of view is categorified by the category of BP-synthetic spectral Syn_{BP}. In particular, there is a self-map of the unit τ causing specializations informally pictured

$$\begin{tikzcd} & \mathrm{Syn}_{BP} \arrow[dl, "\tau=1"'] \arrow[dr, "\tau=0"] & \\ \mathrm{Sp}_{(p)} & & D_{qc}(\mathcal{M}_{\mathrm{fg},(p)}) \end{tikzcd}$$

The category Syn_{BP} itself acts as a sort of bridge for this algebraic-to-stable-homotopy information flow. In particular there is for all $X \in \mathrm{Sp}$ an object $\nu X \in \mathrm{Syn}_{BP}$ whose τ-adic tower

$$X \xleftarrow{\text{invert tau}} \nu X \to \ldots \to \nu X/\tau^n \to \ldots \to \nu X/\tau^2 \to \nu X/\tau$$

encodes the ANSS for X and again mitigates this flow of information from algebraic geometry to stable homotopy theory. For E a nice ring spectrum, we can generalize this phenomenon to the E-Adams spectral sequence. Although the synthetic spectra framework is a nice way to do this, the filtered perspective we saw in Scotty's talk will be our approach for today.

I'll review the theory of filtered spectra and how we use them to "deform" the category of spectra later. For now I want to continue to motivate the main result I will be talking about today. The moduli stack $\mathcal{M}_{\mathrm{fg},(p)}$ has a "height" filtration:

$$\mathcal{M}^{\leq 0}_{\mathrm{fg},(p)} \subset \mathcal{M}^{\leq 1}_{\mathrm{fg},(p)} \subset ...\mathcal{M}^{\leq h}_{\mathrm{fg},(p)} \subset ... \subset \mathcal{M}_{\mathrm{fg},(p)}$$

by formal groups of height $\leq h$. In the same way that $\mathcal{M}_{\mathrm{fg},(p)}$ corresponds to BP, the Morava E-theories $E_{p,h}$ correspond to the filtrations $\mathcal{M}^{\leq h}_{\mathrm{fg},(p)}$. Write $L_h = L_{E_{h,p}}$ for the Bousfield localization by Morava E-theory. Then the Adams-Novikov spectral sequence for $L_h\mathbb{S}$ takes the form:

$$H^*(\mathcal{M}^{\leq h}_{\mathrm{fg},(p)}, \omega^{\otimes i}) \Rightarrow \pi_* L_h\mathbb{S}$$

an important observation is that at large primes this spectral sequence has a sufficiently sparse E_2-page to collapse. This motivates us to ask whether there is some sense in which the entire category $L_h\mathrm{Sp}$ is "algebraic" at large primes. This leads us to the main result we will discuss today:

Theorem 2.1 (We will make this precise later)**.** When $p >> h$ there is a symmetric monoidal equivalence

$$D_{qc}(\mathcal{M}^{\leq h}_{\mathrm{fg},(p)}) \simeq_\alpha L_h\mathrm{Sp}$$

on α-homotopy categories some some α depending on p, h.

Algebraicity results of this form can be very useful. For example, everything on the left hand side can in theory be computed via a chain complex of qc-sheaves on the given moduli stack.

3. Filtered Spectra and Deformations

We will now discuss the formalism which will allow us to build our "bridging" deformations. Let $\mathrm{Pr}^{\mathrm{L,st}}$ be the symmetric monoidal ∞-category of presentable stable ∞-categories and let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L,st}}$. Let $\mathbb{Z}_{\geq}$ denote the integers as a poset category where $m \to n$ exists if $m \geq n$.

Definition 3.1. The category of filtered objects $\mathcal{C}^{\mathrm{fil}}$ is the functor ∞-category

$$\mathcal{C}^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}_{\geq}, \mathcal{C})$$

its objects look like

$$... \to X_1 \to X_0 \to X_{-1} \to ...$$

and maps are given by natural systems of maps $X_i \to Y_i$. We will often exclude any bullet, asterisk, or other indexing symbol when taking a filtered object to avoid notational clutter.

Definition 3.2. The category of filtered objects $\mathcal{C}^{\mathrm{gr}}$ is the functor ∞-category

$$\mathcal{C}^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$$

where $\mathbb{Z}$ is the discrete category.

When working in $\mathrm{Pr}^{\mathrm{L,st}}$, many constructions on an arbitrary $\mathcal{C}$ can be constructed for Sp and brought to $\mathcal{C}$ by tensoring. The above are two examples.

Lemma 3.3. If $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L,st}}$ then $\mathcal{C}^{\mathrm{fil}} \simeq \mathcal{C} \otimes \mathrm{Sp}^{\mathrm{fil}}$ and $\mathcal{C}^{\mathrm{gr}} \simeq \mathcal{C} \otimes \mathrm{Sp}^{\mathrm{gr}}$ where the tensor product is that of $\mathrm{Pr}^{\mathrm{L,st}}$.

This process allows us to prove results for $\mathrm{Sp}^{\mathrm{fil}}, \mathrm{Sp}^{\mathrm{gr}}$ and "tensor them up" to arbitary categories in $\mathrm{Pr}^{\mathrm{L,st}}$. Both of these categories are symmetric monoidal under the day convolution product. As a result, if $\mathcal{C}$ is symmetric monoidal its filtered and graded categories inherit a canonical symmetric monoidal structure from the tensor product. To describe the unit of this object, let $Y : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{fil}}$ denote the functor which takes a spectrum to the filtered object:

$$Y(X) := ... \to 0 \to 0 \to X \to X \to X \to ...$$

where the maps are 0 or identity as appropriate and the first X appears in degree 0.

Lemma 3.4. The unit of the tensor product on $\mathrm{Sp}^{\mathrm{fil}}$ is $Y(\mathbb{S})$. The unit of the tensor product on $\mathrm{Sp}^{\mathrm{gr}}$ is the graded object which is $\mathbb{S}$ in degree 0 and 0 elsewhere.

Remark 3.5. The functor Y is the unit for the symmetric monoidal ∞-category $\mathrm{Sp}^{\mathrm{fil}}$ as an algebra over Sp, i.e. it is the unit symmetric monoidal left adjoint $\mathrm{Sp} \to \mathrm{Sp}^{\mathrm{fil}}$.

There is an important functor:

$$\mathrm{gr} : \mathrm{Sp}^{\mathrm{fil}} \to \mathrm{Sp}^{\mathrm{gr}}$$

defined by $\mathrm{gr}_q X_\bullet = \mathrm{cof}(X_{q+1} \to X_q)$. In the other direction, any graded object may be considered filtered by inserting 0 maps, we will call this functor ι. Another important construction on filtered spectra is the self map τ on a filtered object given by:

$$\tau : X_\bullet[-1] \to X_\bullet$$

whose components τ_q are the internal maps of X itself. For $X_\bullet \in \mathrm{Sp}^{\mathrm{fil}}$ we will denote by X/τ the cofiber of this self map. An interesting fact is that τ preserves multiplicative structures, so that $Y(\mathbb{S})/\tau$ is a commutative ring. The same follows for the quotients $Y(\mathbb{S})/\tau^n$. For notational simplicity we will denote these as $C\tau^n$.

Lemma 3.6. There is an equivalence of endofunctors $\iota \circ \mathrm{gr} \simeq (-)/\tau$. Moreover, $\mathrm{Sp}^{\mathrm{gr}} \simeq \mathrm{Mod}(C\tau)$.

Another important functor are given by $\mathrm{colim}, \lim : \mathrm{Sp}^{\mathrm{fil}} \to \mathrm{Sp}$ which consider a filtered spectrum as a diagram in Sp and take the colimit or limit. The functor colim in particular has a section $\mathrm{const}(-)$ which takes a spectrum X to the constant diagram at X.

Lemma 3.7. There is an equivalence of endofunctors $\tau^{-1}(-) \simeq \mathrm{const} \circ \mathrm{colim}$.

We will often abuse notation and identify the functors $\mathrm{gr}, \mathrm{colim}$ with $(-)/\tau$ and $\tau^{-1}(-)$ respectively, hoping that it is obvious what we intend the codomain to be in context.

Definition 3.8. A $(1-$parameter) deformation is a $\mathrm{Sp}^{\mathrm{fil}}$-module in $\mathrm{Pr}^{\mathrm{L,st}}$. A symmetric monoidal deformation is a $\mathrm{Sp}^{\mathrm{fil}}$-algebra.

Example 3.9. Every stable $\mathcal{C}$ is a $\mathrm{Pr}^{\mathrm{L,st}}$-module over Sp (because it is the unit). The equivalence $\mathrm{Sp}^{\mathrm{fil}} \otimes \mathcal{C} \simeq \mathcal{C}^{\mathrm{fil}}$ shows that $\mathcal{C}^{\mathrm{fil}}$ is a module over $\mathrm{Sp}^{\mathrm{fil}}$. Similarly, if $\mathcal{C}$ is symmetric monoidal then $\mathcal{C}^{\mathrm{fil}}$ is a symmetric monoidal deformation.

Example 3.10. If $\mathcal{C}$ is a symmetric monoidal deformation and $R \in \mathrm{CAlg}(\mathcal{C})$ then $\mathrm{Mod}(R)$ is also a symmetric monoidal deformation via the composition:

$$\mathrm{Sp}^{\mathrm{fil}} \xrightarrow{\text{unit}} \mathcal{C} \xrightarrow{-\otimes R} \mathrm{Mod}(R)$$

Example 3.11. We will return to the following example often. Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. Then the functor $\tau_{\geq *} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{fil}}$ is lax monoidal and sends R to a commutative ring $\tau_{\geq *}R$. Then the category $\mathrm{Sp}_R := \mathrm{Mod}(\tau_{\geq *}R)$ is a symmetric monoidal deformation. In this case the functor left adjoint of $\Re$, which we will denote Y is given by the constant functor at M and the connective cover has the usual underlying filtered object computed levelwise. As a result the composition $\nu(M)/\tau$ may be interpreted as the object $\pi_* M$ in the heart of $D(\pi_* R)$. This encodes a "universal coefficient" spectral sequence:

$$\mathrm{Ext}_{\pi_* R}(\pi_* M, \pi_* N) \Rightarrow \pi_* \mathrm{Map}_{\mathrm{Mod}(R)}(M, N)$$

And a "kunneth spectral sequence" of the form

$$\mathrm{Tor}_{\pi_* R}(\pi_* M, \pi_* N) \Rightarrow \pi_*(M \otimes_R N)$$

It is worth asking what such a deformation is deforming between. The point is that the functors τ^{-1} and $/\tau$ which act on filtered spectra extend to actions on $\mathcal{C}$ itself. Let $\tau^{-1}\mathrm{Sp}^{\mathrm{fil}}$ denote the full subcategory of $\mathrm{Sp}^{\mathrm{fil}}$ in which τ is an autoequivalence and let $\mathrm{Sp}^{\mathrm{fil}}/\tau$ denote $\mathrm{Mod}(C\tau)$. These are both $\mathrm{Sp}^{\mathrm{fil}}$-algebras in $\mathrm{Pr}^{\mathrm{L,st}}$.

Given any deformation $\mathcal{C}$, we may define $\mathcal{C}/\tau := \mathcal{C} \otimes_{\mathrm{Sp}^{\mathrm{fil}}} \mathrm{Sp}^{\mathrm{fil}}/\tau$ and $\tau^{-1}\mathcal{C}$ analogously. We also get functors:

$$(-)/\tau : \mathcal{C} \to \mathcal{C}/\tau \qquad \text{and} \qquad \tau^{-1}(-) : \mathcal{C} \to \tau^{-1}$$

by tensoring with the obvious functors on $\mathrm{Sp}^{\mathrm{fil}}$. Moreover, all of the above discussion holds just as well for the powers of τ. Put $\mathcal{C}^\wedge_\tau := \lim_n \mathcal{C}/\tau^n$. This category may be equivalently described as the full subcategory of $\mathcal{C}$ on objects X such that $X \simeq \lim_n X/\tau^n$. This category is clearly a localizing subcategory of $\mathcal{C}$ admitting a τ-completion functor:

$$(-)^\wedge_\tau : \mathcal{C} \to \mathcal{C}^\wedge_\tau$$

Example 3.12. With Sp_R as before, we can examine the effect of all these functors on the base ring $\pi_{\geq *}R$. It is easy to see that $\tau^{-1}R$ will be $\mathrm{const}(R) \in \mathrm{Sp}^{\mathrm{fil}}$. If we compute $\pi_{\geq *}R \otimes C\tau^m$ we will see in each filtered degree k the cofiber:

$$\tau_{\geq k}R \to \tau_{\geq k-m}R \to \tau_{[k-m,m]}R = (R/\tau^m)_k$$

and we will denote this object by $\tau_{[*,*+m]}R$.

Putting all of this discussion together, we get a τ-adic tower:

$$\tau^{-1}\mathcal{C} \leftarrow \mathcal{C} \to \mathcal{C}^{\wedge}_{\tau} \to \ldots \to \mathcal{C}/\tau^n \to .. \to \mathcal{C}/\tau^2 \to \mathcal{C}/\tau$$

Supposing that we have described $\mathcal{C}/\tau$ as some algebraic category, we might try to prove objects in $\tau^{-1}\mathcal{C}$ are algebraic by lifting them up this tower. Indeed, each successive map in the τ-adic tower turns out to be a square zero extension. The reader unfamiliar with the theory of square zero extensions can get away with following result about lifting modules from $C\tau^n$ to $C\tau^{n+1}$

Lemma 3.13. Let $X \in \mathrm{Mod}(C\tau^n)$. Then the space of lifts of X to $\mathrm{Mod}(C\tau^{n+1})$ can be described via the pullback:

$$\begin{tikzcd} \{\Theta_n \simeq 0\} \arrow[r] \arrow[d] & \mathrm{Mod}(C\tau^{n+1}) \arrow[d] \\ \{X\} \arrow[r, hook] & \mathrm{Mod}(C\tau^n) \end{tikzcd}$$

where the top left corner is the space of nullhomotopies (if they exist) of a certain invariant

$$\Theta_n \in \mathrm{Map}(X/\tau, \Sigma^{n+2}X/\tau[-n])$$

in the $C\tau$-module mapping space.

Intuitively, then, a lift of X to $C\tau^{n+1}$ is equivalent to the data of a nullhomotopy of Θ_n. The functor $X \to X/\tau^n$ can be thought of as a forgetful functor which deletes the information past page n of the associated spectral sequence. Therefore, the process of starting with a filtered spectrum X, killing τ, and attempting to recover X via lifting up the τ-adic tower is exactly the process encoded by the spectral sequence itself. Reusing the same trick again, we can tensor everything we just did with an arbitrary symmetric monoidal deformation $\mathcal{C}$ and recover the same obstruction theory for interpolating between the special and generic fibers.

We run into a problem with this naive approach. Namely any object of $\mathcal{C}$ admitting a $\mathcal{C}/\tau$-lift is 0 is $\tau^{-1}\mathcal{C}$. This conceptual problem may be fixed by equipping deformations $\mathcal{C}$ with t-structures. We will discuss this after a final general construction for deformations.

Let $\mathrm{Sp}^{\mathrm{fil}}[\tau = 0]$ be defined to be the category Sp equipped with the $\mathrm{Sp}^{\mathrm{fil}}$-algebra structure given by the composition:

$$\mathrm{Sp}^{\mathrm{fil}} \xrightarrow{\mathrm{gr}} \mathrm{Sp}^{\mathrm{gr}} \xrightarrow{\mathrm{colim}} \mathrm{Sp}$$

and if $\mathcal{C}$ is some other deformation, define $\mathcal{C}[\tau = 0] := \mathcal{C} \otimes_{\mathrm{Sp}^{\mathrm{fil}}} \mathrm{Sp}^{\mathrm{fil}}[\tau = 0]$. We of course get functors $(-)[\tau = 0]$ as usual.

Remark 3.14. We could analogously define $\mathrm{Sp}^{\mathrm{fil}}[\tau = 1]$ to be the algebra structure on Sp given by

$$\mathrm{Sp}^{\mathrm{fil}} \to \tau^{-1}\mathrm{Sp}^{\mathrm{fil}} \xrightarrow{\mathrm{colim}} \mathrm{Sp}$$

and $\mathcal{C}[\tau = 1]$ via tensorsing. This is unecessary, however, as the second map is an equivalence with inverse const. The same is of course not true for $[\tau = 0]$ and $/\tau$ since $\mathrm{Sp} \neq \mathrm{Sp}^{\mathrm{gr}}$.

Example 3.15. For Sp_R, we have that $\tau_{\geq *}R[\tau = 0] = H\pi_*R := \bigoplus \pi_i R[i]$ and $\tau^{-1}\tau_{\geq *}R \simeq R$ when we identify constant filtered spectra with spectra. In fact we may identify all of the auxilary categories so far defined:

- $\mathrm{Sp}_R/\tau \simeq \mathrm{Mod}(\tau_{=*}R)$ modules of graded spectra over the graded spectrum which is $\pi_i R[i]$ in degree i.
- $\mathrm{Sp}_R[\tau = 0] \simeq \mathrm{Mod}(H\pi_*R) \simeq D(\pi_*R)$
- $\tau^{-1}\mathrm{Sp}_R \simeq \mathrm{Mod}(R)$

A cute version of this is when $R \in \mathrm{Sp}^{\heartsuit} \simeq \mathrm{Ab}$ is a classical commutative ring. In this case we can prove a toy algebraicity result $\tau^{-1}\mathrm{Sp}_R \simeq \mathrm{Sp}_R[\tau = 0] \simeq D(R)$.

4. Goerss-Hopkins Deformations

In order to motivate the definitions of this section, I am going to review some facts about the category of synthetic spectra. The functor $\Re : \mathrm{Syn}_E \to \mathrm{Sp}$ preserves colimits, and hence by the adjoint functor theorem admits a right adjoint Y. Let $\tau_{\geq n} : \mathrm{Syn}_E \to (\mathrm{Syn}_E)_{\geq n}$ denote the truncation for the t-structure considered in Piotr's paper. This is constructed as the t-structure associated to viewing Syn_E as a category of sheaves. We recall some salient facts about Y:

Proposition 4.1. The functor $Y : \mathrm{Sp} \to \mathrm{Syn}_E$ is fully faithful and the functor and realizes Sp as the subcategory of τ-inverted synthetic spectra so that the realization functor $\mathrm{Syn}_E \to \mathrm{Sp}$ factors as:

$$\Re : \mathrm{Syn}_E \xrightarrow{\tau^{-1}} \tau^{-1}\mathrm{Syn}_E \xleftarrow{\simeq} \mathrm{Sp}$$

Moreover, there is an equivalence $\nu X \simeq \tau_{\geq 0} Y(X)$.

If we take the t-structure and $\Re$ for granted, one could argue that the above provides a conceptually clear *definition* of $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$. This is the perspective we will take in defining a Goerss-Hopkins deformation. First, as with everything we've done so far, we will discuss the example of $\mathrm{Sp}^{\mathrm{fil}}$.

Definition 4.2. The diagonal t-structure on $\mathrm{Sp}^{\mathrm{fil}}$ is defined by:

$$\mathrm{Sp}^{\mathrm{fil}}_{\geq n} = \{X \in \mathrm{Sp}^{\mathrm{fil}} \mid X_i \in \mathrm{Sp}_{\geq i+n}\}$$
$$\mathrm{Sp}^{\mathrm{fil}}_{\leq n} = \{X \in \mathrm{Sp}^{\mathrm{fil}} \mid X_i \in \mathrm{Sp}_{\leq i+n}\}$$

where Sp has its usual t-structure.

Remark 4.3. We note that if $X \in \mathrm{Sp}^{\mathrm{fil}}$ then the associated truncations $\tau_{\geq n} X$ are computed levelwise as

$$...\tau_{\geq n+1} X_1 \to \tau_{\geq n} X_0 \to \tau_{\geq n-1} X_{-1} \to ...$$

so that $\tau_{\geq 0}(\mathrm{const}(Y))$ recovers the Postnikov tower of a spectrum Y.

Definition 4.4. A Goerss Hopkins deformation $\mathcal{E}$ is a symmetric monoidal deformation equipped with a t-structure satisfying:

(1) $\bigcap_n \mathcal{E}_{\leq n} = \{0\}$ (any infinitely coconnective object is 0)
(2) $\tau_{\geq 0} : \mathcal{E} \to \mathcal{E}_{\geq 0}$ preserves filtered colimits
(3) For all $X \in \mathrm{Sp}^{\mathrm{fil}}_{\geq 0}$ and $E \in \mathcal{E}_{\geq 0}$ we have $X \otimes E \in \mathcal{E}_{\geq 0}$.[1]

In this setting, we already have the realization functor[2]:

$$\Re : \mathcal{E} \to \tau^{-1}\mathcal{E}$$

which will admit a left adjoint, allowing us to define a functor $\nu : \tau^{-1}\mathcal{E} \to \mathcal{E}$ as the connective cover of the adjoint to $\Re$.

Example 4.5. Let Sp_R be modules over the postnikov tower of R as before. We can put:

$$(\mathrm{Sp}_R)_{\geq 0} := \mathrm{Mod}(\mathrm{Sp}^{\mathrm{fil}}_{\geq 0}, \tau_{\geq *} R)$$

and extend this to a t-structure on Sp_R which satisfies the definition of a Goerss-Hopkins deformation.

The primary example we care about will be a modification of the above idea. In order to connect it to the talk Scotty gave before this, I am going to take a slight digression.

Definition 4.6. Let R be a commutative ring in any symmetric monoidal category $\mathcal{C}$. Then it has a cobar complex $\mathrm{cb}_R := R^{\bullet+1}$ which is a cosimplicial spectrum. This compiles into a functor $\mathrm{cb}_R(-) : \mathcal{C} \to \mathcal{C}^\Delta$ via $X \mapsto X \otimes \mathrm{cb}_R$.

Note tensor products of rings are rings so that each $\mathrm{cb}^k_R = R^{k+1}$ is again a ring. We can apply the postnikov tower functor to get an object:

$$\Gamma^R_{n,k} = \tau_{\geq n}(R^{k+1})$$

Taken altogether this is an object in $\mathrm{Fun}(\Delta \times \mathbb{Z}_{\geq}, \mathrm{Sp})$ and we can think of it either as a filtered cosimplicial spectrum or a cosimplicial filtered spectrum. In either case we can consider the totalization. If we think of it as a cosimplicial filtered spectrum we see:

$$\Gamma^R := \mathrm{Tot}_\bullet \left(\tau_{\geq *} R^{\bullet+1}\right)$$

however, because limits in a functor category are computed levelwise at each filtered level this looks like

$$\Gamma^R_n = \mathrm{Tot}_\bullet \left(\tau_{\geq n} R^{\bullet+1}\right)$$

[1]Recall that the symmetric monoidal unit $c : \mathrm{Sp}^{\mathrm{fil}} \to \mathcal{E}$ allows us to consider $\mathcal{E}$ to be "tensored over" $\mathrm{Sp}^{\mathrm{fil}}$ via the formula $X \otimes E := c(X) \otimes E$.
[2]I am abusing notation a bit here since previously I used the $\tau^{-1}\mathrm{Syn}_E$ notation to mean the τ-invertible subcategory and here I mean the special fiber.

recovering the notation of GIKR up to this evenness difference.[3] This can be extended to a lax monoidal functor

$$\Gamma^R_* \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{fil}}$$
$$X \mapsto \mathrm{Tot}\left(\tau_{\geq *}(X \otimes \mathrm{cb}_R)\right)$$

Remark 4.7. I think that some descent-type argument should be able to show an equivalence

$$\mathrm{Mod}(\Gamma^R_* \mathbb{S}) \simeq \mathrm{Tot}\,\mathrm{Sp}_{R^{k+1}}$$

but I could not work out the details, at least not in time for the talk. If anyone knows how to argue this (it should be related to work of Mathew on descent) or why it is not quite true, please let me know!

We know that the category of modules over $\Gamma^R\mathbb{S}$ is of essential importance as it encodes the E-Adams-Novikov. We would like to be able to view it as a Goerss-Hopkins deformation (it is already a deformation by construction). We will do it for the categorical totalization instead.

Lemma 4.8. Let $R(-) : J \to \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ be a diagram of spectra in which all the maps are π_*-flat, i.e. each $\pi_* R(j) \to \pi_* R(j')$ is a flat morphism of graded rings. Put $\mathcal{E}_\infty := \lim_j \mathrm{Sp}_{R(j)}$. Then $\mathcal{E}_\infty$ is a Goerss Hopkins deformation such that all of whose auxiliary categories are the limits of those for the $\mathrm{Sp}_{R(j)}$.

For a more precise version of the above see the original paper's Prop 2.32. We will be interested specifically in the deformations associated as the above to the case $J = \Delta$, i.e., cosimplicial rings.

Lemma 4.9. Let A be a ring commutative spectrum such that A_*A is A_*-flat. Then

$$\mathcal{E}_R := \mathrm{Tot}_k\,\mathrm{Sp}_{R^{k+1}}$$

is a Goerss-Hopkins deformation subject to the above lemma, i.e., we can describe its various auxiliary categories in terms of those for the $\mathrm{Sp}_{R^{k+1}}$.

Proof. The trick here is to replace Δ by the initial subcategory Δ^{inj} and set this to be J in the above lemma. Then the flatness assumption guarantees that at each injective morphism is sent to one that is π_*-flat. □

The main category of interest which will relate chromatic homotopy theory to an algebraic category is

$$\mathcal{E}_{p,h} := \mathcal{E}_{E_{p,h}}$$

where $E_{p,h}$ is the Lubin tate theory of the height h Honda formal group law at the prime p. We can read off that is has the following specializations:

Lemma 4.10. The specializations of $E_{p,h}$ are the following

- We always have $\tau^{-1}\mathcal{E}_{p,h} \simeq \mathrm{Sp}_{p,h} := L_{E_{p,h}}\mathrm{Sp}$.
- When $p > h + 1$ we have $\mathcal{E}_{p,h}[\tau = 0] \simeq \mathrm{Fr}_{p,h} := \mathrm{Tot}\,\mathrm{Sp}_{H(\pi_* R^{k+1})}$.[4]
- When $p > h + 1$ we have $\mathcal{E}_{p,h}/\tau \simeq D(\mathrm{grComod}(E_*E))$.

where the hypothesis $p > h + 1$ is used to relate $\mathcal{E}_{p,h}$ to $D(\mathrm{grComod}(E_*E))$ based on their equivalent hearts.

5. Goerss-Hopkins Tower

We now can discuss the corrected version of the τ-adic tower. Note first that because τ acts in paralell to the diagonal t-structure, the functor:

$$\tau^{-1}\mathcal{E} \xrightarrow{\tau_{\geq 0}} \mathcal{E}_{\geq 0}$$

is fully faithful with inverse given by simply re-inverting τ. Not also that the inclusion of the heart $\mathcal{E}^{\heartsuit} \hookrightarrow \mathcal{E}$ factors through $\mathcal{E}/\tau$ for the same reason. As a result, the following is a symmetric monoidal pullback diagram:

$$\begin{array}{ccc} \tau^{-1}\mathcal{E} & \longrightarrow & \mathcal{E}_{\geq 0} \\ \downarrow & & \downarrow \\ \mathcal{E}^{\heartsuit} & \longrightarrow & \mathcal{E}/\tau \end{array}$$

[3] This difference ends up being not so substantial, the resulting deformation where you replace $\tau_{\geq *}$ by $\tau_{\geq 2*}$ ends up being a subcategory generated by "even" objects.

[4] Shaul puts the $p > h + 1$ hypothesis here but I cannot tell where he uses it.

which encourages us to attempt to pull back the entire tower in such a manner:

$$\begin{tikzcd} & \dots \arrow[d] \\ \mathcal{M}_n\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}/\tau^n \arrow[d] \\ \mathcal{E}^{\heartsuit} \arrow[r] & \mathcal{E}/\tau^{n-1} \arrow[d] \\ & \dots \end{tikzcd}$$

where the category $\mathcal{M}_n\mathcal{E}$ is defined to be the pullback. We define the Goerss-Hopkins tower to be:

$$\tau^{-1}\mathcal{E} \to \lim_k \mathcal{M}_k\mathcal{E} \to ... \to \mathcal{M}_k\mathcal{E} \to ... \to \mathcal{M}_1\mathcal{E} \to \mathcal{M}_0\mathcal{E} \simeq \mathcal{E}^{\heartsuit}$$

where the first map is the universal one. Our goal will be to use this tower to compare $\tau^{-1}\mathcal{E}$ with $\mathcal{E}/\tau$. We can think of the functors $\tau^{-1}\mathcal{E} \to \mathcal{M}_k\mathcal{E}$ as being given by first using truncation to "uninvert" τ and only *then* take $(-)/\tau^{k+1}$. Algebraicly this can be viewed as taking a graded module where τ acts invertibly with degree 1, and using the fact that the module is graded to get rid of the elements $\frac{x}{\tau}$ for $|x| = 0$.

Remark 5.1. For the comparison with usual Goerss-Hopkins theory, see VanKoughnett-Pstragowski.

The subcategories $\mathcal{M}_k\mathcal{E}$ admit a more direct description. Let $X \in \mathcal{E}_{\geq 0}/\tau^{n+1}$.

Definition 5.2. With notation as above, X is a potential n-stage if the maps:

$$X/\tau^{k+1} \to \tau_{\geq k}X$$

is an equivalence for all $k \leq n$. This turns out to be equivalent to asking that $X/\tau \simeq \tau_{\geq 0}X$.

Lemma 5.3. The subcategory $\mathcal{M}_k\mathcal{E}$ may be described as the subcategory of $\mathcal{E}_{\geq 0}/\tau^{k+1}$ consisting of the potential k-stages.

Intuitively, this should be thought of as follows. Objects in the image of $\tau^{-1}\mathcal{E} \to \mathcal{E}_{\geq 0}$ are inherently polynomial on τ, i.e., τ induces a degree 1 equivalence on each of the homotopy groups. As such, killing a tower of τ will look like an object with the same homotopy in each degree up to the power killed, and then nothing in higher degrees, i.e., the described truncations. Explicitly X is a potential k-stage if and only if

$$\pi_*X = \pi_0X[\tau]/\tau^{k+1}$$

Finally we end this section with a convergence statement:

Proposition 5.4. If $\mathcal{E}$ is complete in the sense that its connective part is Postnikov complete, then the map $\tau^{-1}\mathcal{E} \to \lim_k \mathcal{M}_k\mathcal{E}$ is an equivalence.

6. Degeneracy Structures

Let $X \in \mathrm{Sp}^{\mathrm{fil}}$ be a filtered spectrum. When attempting to reconstruct $\tau^{-1}X$ from gr_*X, the dumbest possible thing that could happen is that

$$\tau^{-1}X \simeq \bigoplus \mathrm{gr}_*X$$

and $X_n = \bigoplus_{i \geq n} \mathrm{gr}_iX$. Such a filtered object is said to be *degenerate*. There is a functor:

$$\delta : \mathrm{Sp}^{\mathrm{fil}} \to \mathrm{Sp}^{\mathrm{fil}}$$

which sends X to the degeneration of X given by the formula

$$\delta(X)_n = \bigoplus_{i \geq n} \mathrm{gr}_iX$$

which will have the same associated graded as X but no differentials in its associated spectral sequence. Explicitly if $X/\tau^n \simeq \delta(X)/\tau^n$ as $C\tau^n$-modules then the associated spectral sequence has no nonzero differentials before page n.

The functor $\delta : \mathrm{Sp}^{\mathrm{fil}} \to \mathrm{Sp}^{\mathrm{fil}}$ is symmetric monoidal and hence once again presents $\mathrm{Sp}^{\mathrm{fil}}$ as a commutative algebra over itself which we will denote $\mathrm{Sp}^{\mathrm{fil},\delta}$.

Definition 6.1. For a deformation $\mathcal{C}$ let $\mathcal{C}^\delta := \mathcal{C} \otimes_{\mathrm{Sp}^{\mathrm{fil}}} \mathrm{Sp}^{\mathrm{fil},\delta}$.

Lemma 6.2. Let $\mathcal{E}$ be a symmetric monoidal Goerss-Hopkins deformation. Then $\mathcal{E}^\delta$ is also a symmetric monoidal Goerss-Hopkins deformation.

Let us write $H\pi_* R := \bigoplus_i \Sigma^i H\pi_i R$.

Example 6.3. We can explicitly describe the degeneracy of the categories Sp_R explicitly as $\mathrm{Sp}_{H\pi_* R}$.

We can finally state the main definition in the study of degenerate filtrations and deformations:

Definition 6.4. An m-degeneracy structure on a symmetric monoidal Goerss-Hopkins deformation $\mathcal{E}$ is a $\mathrm{Sp}^{\mathrm{fil}}/\tau^{m+1}$-linear symmetric monoidal equivalence:

$$\mathcal{E}/\tau^{m+1} \simeq \mathcal{E}^\delta/\tau^{m+1}$$

Roughly speaking the larger we can take m and still find such an equivalence, the more trivial the spectral sequence in question is. In the extreme, we see that the spectral sequence collapses completely demonstrating the complete "algebraicity" of the deformation.

Example 6.5. Let Sp_R be the category of modules over the Postnikov filtration of R as before. Then the space of all m-degeneracy structures on R is equivalent to the space of equivalences of commutative $C\tau^{m+1}$-algebras $\tau_{*,*+m} R \simeq \tau_{*,*+m} H\pi_* R$.

Let $\mathcal{M}_k\mathcal{E}$ be the Goerss-Hopkins tower from before. Then given an m-degeneracy structure

$$\gamma : \mathcal{E}/\tau^{m+1} \xrightarrow{\simeq} \mathcal{E}^\delta/\tau^{m+1}$$

the restriction induces an equivalence $\mathcal{M}_m\mathcal{E} \simeq \mathcal{M}_m\mathcal{E}^\delta$. The general theorem that is proven in this paper is the following:

Theorem 6.6 (Algebraicity Theorem)**.** Let $\mathcal{E}$ be a symmetric monoidal Goerss-Hopkins deformation such that:

(1) The connective part is postnikov complete ($\mathcal{E}_{\geq 0} \simeq \lim \mathcal{E}_{\leq n}$).
(2) $\mathcal{E}/\tau$ is generated under colimits by the discrete objects.
(3) $\mathcal{E}^\heartsuit$ has finite Ext and Tor dimension e, d respectively.

Then an m-degeneracy structure on $\mathcal{E}$ induces a symmetric monoidal equivalence:

$$\tau^{-1}\mathcal{E} \simeq_{\lfloor \frac{m-d+4}{e+1} - 3 \rfloor} \mathcal{E}/\tau$$

on homotopy $\lfloor \frac{m-d+4}{e+1} - 3 \rfloor$-categories.

Remark 6.7. It is worth noting that the level of equivalence decreases linearly in the bound on Tor dimension, but is inversely proportional to Ext dimension.

Once we establish the above, all that will remain (aside from working out the numerics) is to establish a degeneracy structure on the category $\mathcal{E}_{p,h}$. This can in fact be done in much greater generality.

Proposition 6.8. Let $R(-) : J \to \mathrm{CAlg}(\mathrm{Sp})$ be as before a diagram of connective ring spectra and π_*-flat maps between them. Then if the homotopy groups of $R(j)$ are concentrated in degrees divisible by m. Then there exists an m-degeneracy structure on $\mathcal{E}_\infty$.

Proof. We explain the case $J = pt$ and note that all of this can be done functorially in J if it is larger. First note that an m-degeneracy structure $\gamma : \mathrm{Sp}_R/\tau^{m+1} \simeq \mathrm{Sp}_{H\pi_* R}/\tau^{m+1}$ is equivalent data to an equivalence of rings $\tau_{[*,*+m]} R \simeq \tau_{[*,*+m]} H\pi_* R$. Via some adjunction juggling we get an equivalence:

$$\mathrm{Map}_{E_\infty, \mathrm{Sp}^{\mathrm{fil}}/\tau^{m+1}}(\tau_{[*,*+m]} H\pi_* R, \tau_{[*,*+m]} R) \simeq \mathrm{Map}_{E_\infty, \mathrm{Sp}^{\mathrm{gr}}}(\tau_{=*} R, \iota^*(\tau_{[*,*+m]} R))$$

and that something on the left is an equivalence if and only if the associated map on the right defines a section of the truncation $\iota^*\tau_{[*,*+m]} R \to \tau_{=*} R$.

To show that such a section exists, we will induct on $j \leq m$. The case $j = 0$ is the identity. Then we note that $\iota^*\tau_{[*,*+j+1]} R \to \iota^*\tau_{[*,*+j]} R$ is a square zero extension as it correspond to $C\tau^{j+1} \to C\tau^j$. By a deformation theory argument we have a fiber sequence:

$$\begin{tikzcd}
\mathrm{Sect}(\iota^*\tau_{[*,*+j+1]}R \to \tau_{=*}R) \arrow[r] \arrow[d] & \mathrm{Sect}(\iota^*\tau_{[*,*+j]}R \to \tau_{=*}R) \arrow[d] \\
pt \arrow[r] & \mathrm{Map}_{\mathrm{Mod}(\tau_{=*}R)}(L_{\tau_{=*}R}, \Sigma^{j+2}\tau_{=*+j+1}R)
\end{tikzcd}$$

and this last space vanishes for degree reasons as long as $j+1$ is not 0 mod $m+1$. $\square$

7. Obstruction Theory

As alluded to before, each or the maps $C\tau^{n+1} \to C\tau^n$ is a square zero extension. Without getting into the theory of such, it turns out that lifts of $C\tau^n$-modules along the projection are controlled by the triviality of a certain map

$$\theta_n : C\tau^n \to \Sigma^{n+2}C\tau[-n]$$

a central feature of much obstruction theory is that the construction of this map is largely irrelevant as essentially the only standard method of showing it vanishes is to show that the whole space of maps is trivial. Letting $\mathcal{E}$ be a fixed symmetric monoidal Goerss-Hopkins deformation. After tensoring with an object $X \in \mathcal{E}/\tau^n$ we have a pullback

$$\begin{tikzcd} \{\theta_n \otimes X \simeq 0\} \arrow[r] \arrow[d] & \mathcal{E}/\tau^{n+1} \arrow[d] \\ \{X\} \arrow[r, hook] & \mathcal{E}/\tau^n \end{tikzcd}$$

showing that the space of lifts of X to $\mathcal{E}/\tau^{n+1}$ is controlled by the space of nullhomotopies of $\theta_n \otimes X : X \to \Sigma^{n+3}X/\tau[-n-1]$. In particular, the existence of any nullhomotopy guarantees a lift of X. Our goal is to use this to create an obstruction theory for the Goerss-Hopkins tower $\mathcal{M}_n\mathcal{E}$.

Proposition 7.1. Let $X \in \mathcal{M}_n\mathcal{E}$ be a potential n-stage. Then there exists an element

$$o_X \in \mathrm{Ext}^{n+3,n+1}_{\mathcal{E}/\tau}(\pi_0X, \pi_0X)$$

such that there exists a lift of X to $\mathcal{M}_{n+1}\mathcal{E}$ if and only if $o_X = 0$.

Proof. Taking the existence of the obstruction theory above for granted, we observe that since X is assumed to be a potential n-stage we have

$$\mathrm{Map}_{\mathcal{E}/\tau^{n+1}}(X, \Sigma^{n+3}X/\tau[-n-1]) \simeq \mathrm{Map}_{\mathcal{E}/\tau}(X/\tau, \Sigma^{n+3}X/\tau[-n-1]) \simeq \mathrm{Map}_{\mathcal{E}/\tau}(\pi_0X, \Sigma^{n+3}\pi_0X[-n-1])$$

$\square$

We now return to the algebraicity theorem. In order to exhibit Shaul's full result, I do need to pay attention to model structures here.

Lemma 7.2. Let $\mathcal{E}$ be a Goerss-Hopkins deformation with enough flat objects, finite Tor-dimension $\leq e$ and Ext-dimesion $\leq d$ such that $\mathcal{E}/\tau$ is generated by discretes. Then the functor:

$$\tau^{-1}\mathcal{E}^{\otimes} \to \mathcal{M}_m\mathcal{E}^{\otimes}$$

is essentially surjective on objects whenever $m \geq d$ it also induces an $(m-d-(r-1)e)$-equivalence on the arity r multimapping spaces of the operads.

Proof. We will work inductively showing that all of the maps $\mathcal{M}_{m+1}\mathcal{E} \to \mathcal{M}_m\mathcal{E}$ and above satisfy the claim, since the tower converges. On objects the claim is clear as the obstructions to lifting to a potential $m+1$-stage live above the assumed Ext-dimension.

We will only concern ourselves with the E_∞-version of the theory. Shaul shows analogously to our square zero theory for objects that there is a fiber sequence of spaces

$$\mathrm{Map}_{\mathcal{M}_{m+1}\mathcal{E}}(X^1 \otimes \ldots \otimes X^r, Y) \to \mathrm{Map}_{\mathcal{M}_m\mathcal{E}}(\tau_{\leq m}X^1 \otimes \ldots \otimes \tau_{\leq m}X^r, Y) \to \mathrm{Map}_{\mathcal{E}/\tau}(\pi_0X^1 \otimes \ldots \otimes \pi_0X^r, \Sigma^{m+2}Y[-m-1])$$

So that it suffices to show that $\mathrm{Map}_{\mathcal{E}/\tau}(\pi_0X^1 \otimes \ldots \otimes \pi_0X^r, \Sigma^{m+2}Y[-m-1])$ has connectivity $(m+1-d-(r-1)e)$. The flat objects assumption is used in showing that $\mathcal{E}/\tau$ has the same Tor and Ext bounds as the heart. From there we use both the Ext and Tor bounds to establish the needed connectivity of this mapping space. $\square$

Proof of Theorem 6.6. Let us recall our assumptions. Let $\mathcal{E}$ be a symmetric monoidal Goerss-Hopkins deformation whose connective part is postnikov complete, whose $\mathcal{E}/\tau$ is generated by discrete objects, and the discrete objects have Ext, Tor dimension bounds e, d.

Then we assume we are given an m-degeneracy structure:

$$\gamma : \mathcal{E}/\tau^{m+1} \simeq \mathcal{E}^{\delta}/\tau^m$$

this restricts to potential m-stages as an equivalence $\mathcal{M}_m\mathcal{E} \simeq \mathcal{M}_m\mathcal{E}^\delta$. Moreover, regardless of our assumptions we can see that there is always an equivalence $\mathcal{E}/\tau \simeq \mathcal{E}^\delta/\tau$ so that $\mathcal{E}^\delta$ satisfies the same generation and dimensional assumptions. Let $\alpha = \lfloor \frac{m-d+4}{e+1} - 3 \rfloor$. It suffices to show that

$$\tau^{-1}\mathcal{E} \to \mathcal{M}_m\mathcal{E}$$

is an equivalence on homotopy α-categories, as then we have such an equivalence between $\tau^{-1}\mathcal{E}$ and $\mathcal{M}_m\mathcal{E}^\delta$. But our potential m stages are truncated polynomial on their π_0, so that $[\tau = 0]$ is an equivalence. After appealing to the above lemma, we are done. $\square$

8. Application to Chromatic Homotopy Theory

Recall the construction from before of the categories:

$$\mathcal{E}_R := \operatorname{Tot} \operatorname{Sp}_{R^{k+1}}$$

Let E be height h Morave E-theory at the prime p. This ring spectrum carries an action of $\mathbb{F}_p^\times$. Put $A = E^{h\mathbb{F}_p^\times}$. Earlier I claimed we wanted to study the category $\mathcal{E}_E$. We will instead study $\mathcal{E}_A$. The point is that E splits as sums of shifts of A and that A as a result has better periodicity, i.e., it is $2p-2$-periodic.

Theorem 8.1. Whenever $p > \frac{1}{2}h^2 + \frac{k+3}{2}h + \frac{k+1}{2}$ we have a symmetric monoidal equivalence:

$$\mathrm{Fr}_{p,h} \simeq_k L_{E_{p,h}}\mathrm{Sp}$$

on homotopy k-categories. Moreover, this is a tensor-triangulated equivalence on homotopy 1-categories.

Proof. Recall that the deformation $\mathcal{E}_{p,h}$ has specializations

$$\begin{tikzcd} & \mathcal{E}_{E_{p,h}} \arrow[dl, "\tau=0"'] \arrow[dr, "\tau=1"] & \\ \mathrm{Fr}_{p,h} & & L_{E_{p,h}}\mathrm{Sp} \end{tikzcd}$$

Moreover if $p > h+1$ there is a t-exact symmetric monoidal equivalence $\mathrm{Mod}_{\mathcal{E}_{p,h}}(C\tau) \simeq D(\mathrm{grComod}(E_*E))$. The relevant comodule facts follow from work of Hovey. The degeneracy structure follows from the to-be-described obstruction theory and the fact that A is $2p-2$-periodic. The numerics work out (Honestly I didn't check them). $\square$